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AN APPROXIMATE ANALYSIS OF THE SHOCK STRUCTURE IN UNDEREXPANDED PLUMES

ENGINE TEST FACILITY
ARNOLD ENGINEERING DEVELOPMENT CENTER
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Prepared for

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An approximate analytical model has been developed for computing the first cycle of the shock structure in plumes from underexpanded axisymmetric supersonic nozzles. The physical basis of the model is the same as that of the Abbett model, i.e., the Mach disk location is determined by the coupling of the flow in the vicinity of the Mach disk with the reaccelerating transonic flow downstream of the disk. ~~However, instead of the Abbett~~

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20. ABSTRACT (Continued)

~~technique of iterative flow-field computations between the trial Mach disk and the transonic region,~~ the present model is based on a local compatibility condition for the flow just downstream of the Mach disk. To evaluate the local compatibility condition, the generalized shape of the slip line which originates at the shock triple point must be known. The Abbett method was used to compute slip line geometries for a number of plume flows; correlation equations for the slip line geometrical parameters were then developed. ~~The flow field upstream of the Mach disk is computed with the rotational method of characteristics. At each point on the boundary shock wave, a trial triple point is assumed and the local compatibility condition is evaluated; the Mach disk is located where the compatibility equation is satisfied.~~ After the Mach disk is located, the supersonic flow outside of the slip line is computed for the prescribed slip line geometry. The approximate analytical model requires relatively small computation times and yields acceptably accurate flow-field solutions over the range of conditions for which the slip line correlation equations are valid.

PREFACE

The work reported herein was conducted by the Arnold Engineering Development Center (AEDC), Air Force Systems Command (AFSC), under Program Element 65807F. The results were obtained by ARO, Inc. (a subsidiary of Sverdrup & Parcel and Associates, Inc.), contract operator of AEDC, AFSC, Arnold Air Force Station, Tennessee. The work was done under ARO Project Nos. RF423, R33P-60A, and R33A-02A. The authors of this report were C. E. Peters and W. J. Phares, ARO, Inc.

The Fox-Abbett computations which are discussed in this report were carried out by J. H. Fox of the Central Data Processing Division of ARO, Inc. The manuscript (ARO Control No. ARO-ETF-TR-76-85) was submitted for publication on August 5, 1976.

CONTENTS

ILLUSTRATIONS

1.0 INTRODUCTION

Numerous experimental investigations (e.g., Refs. 1 through 4) have defined the general structure of the axisymmetric plume which is formed when the flow from an underexpanded sonic or supersonic nozzle exhausts into a quiescent ambient medium (Fig. 1). A boundary shock wave is formed by the coalescence of compression waves from the curved jet boundary. As the boundary shock wave propagates downstream, it becomes stronger and curves toward the axis. For slightly underexpanded nozzle operation ($p_n/p_\infty < 1.5$), the boundary shock wave appears to undergo a regular reflection at the axis. As p_n/p_∞ is increased, however, the boundary shock no longer undergoes a regular reflection at the axis; instead, a nearly normal shock wave, called the "Mach disk" or "Riemann wave", occurs (Fig. 1). The boundary shock, the Mach disk and the reflected shock are joined at a triple point.

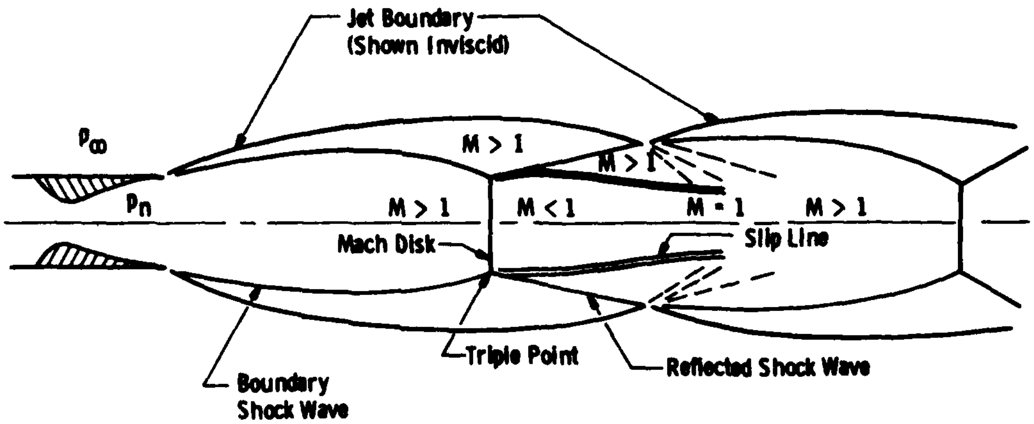


Figure 1. Multiple shock wave structure in plumes.

For moderately underexpanded nozzle operation (p_n/p_∞ up to 10 or 20), the plume shock structure has been observed to repeat with increasing distance (Fig. 1). Thus, the flow behind the Mach disk reaccelerates to supersonic speeds then passes through another shock system. The observed number of repetitions of the shock system depends on p_n/p_∞, but the

inviscid shock wave structure is usually obliterated after two or three cycles by the turbulent mixing layer between the jet and the ambient fluid.

If p_n/p_∞ is very large, perhaps 100, then the maximum diameter of the plume and the diameter of the Mach disk are much larger than the nozzle exit diameter. It is not clear from the available experiments whether the plume shock structure is repetitive at such high pressure ratios. Almost all of the experimental information on plume shock structure has been obtained by optical methods; for highly underexpanded plumes, the shock structure downstream of the first Mach disk is obscured by the turbulent mixing along the plume boundary.

Various approximate techniques have been proposed for predicting the location of the first Mach disk. The empirical correlation equation of Lewis and Carlson (Ref. 5) yields acceptably accurate Mach disk positions, but the technique provides no information about the plume structure other than the Mach disk location. Adamson and Nicholls (Ref. 2) assumed that the Mach disk occurs at a position where the pressure behind the normal shock on the centerline is exactly equal to the ambient pressure, p_∞. This method requires only that the flow properties along the plume centerline be defined. The Mach disk locations predicted with the Adamson and Nicholls method are, in general, less accurate than those predicted with the Lewis and Carlson empirical equation. No information is provided about the two-dimensional distributions of flow properties in the plume with either method.

Several investigators have proposed models for Mach disk location which are based on detailed computation of the plume flow field. The inviscid flow upstream of the Mach disk can be computed with the rotational method of characteristics. However, the method of characteristics offers no insight into the mechanism of Mach disk formation; the flow field, including the boundary shock wave, can be computed downstream beyond the experimental location of the Mach disk, with no indication

that the solution really does not occur. Therefore, in addition to the rotational method of characteristics, the "field" methods require assumptions about the mechanism of Mach disk formation.

Eastman and Radtke (Ref. 6) assumed that the Mach disk occurs at a position where the static pressure behind the computed boundary shock reaches a minimum. Bowyer, et al., (Ref. 7) assumed that the Mach disk is a normal shock at the triple point. Therefore, the Mach disk is predicted to occur at the station where the usual triple point conditions ($p_3 = p_4$, $\theta_3 = \theta_4$) are satisfied and where $\theta_1 = \theta_3$ (Fig. 2).

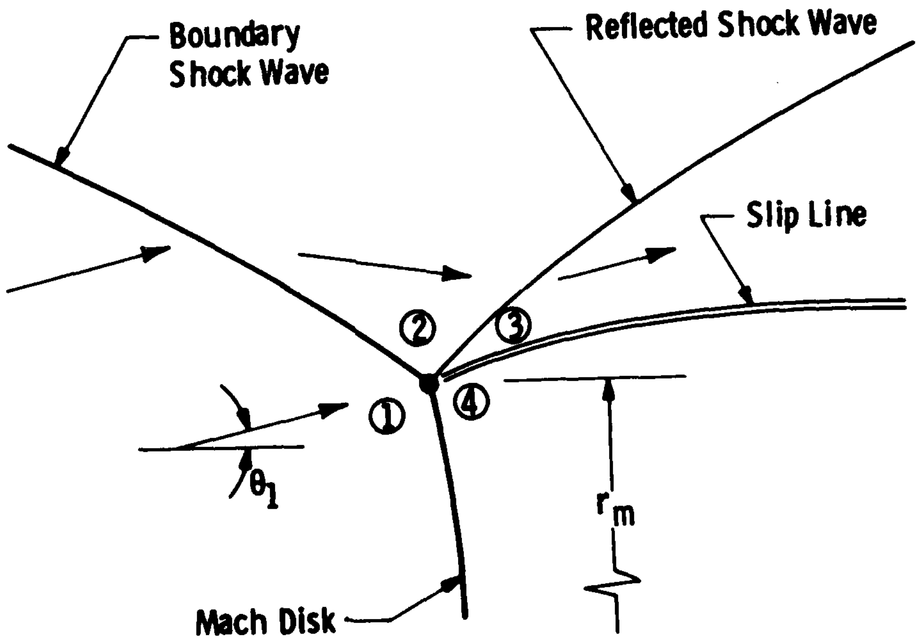


Figure 2. Flow in vicinity of the shock triple point.

The Eastman and Radtke and Bowyer, et al., field methods are attractive because they provide a prediction of the entire plume flow field upstream of the Mach disk. But these methods predict Mach disk locations which are in only fair agreement with experiment (Ref. 8). Furthermore,

the assumed mechanisms of Mach disk formation do not have a sound physical basis.

The "field" methods were generally unsuccessful until a physically perceptive mechanism for Mach disk formation was postulated. The correct physical mechanism was pointed out by Peters (Ref. 9), Peters and Phares (Ref. 8), and Abbett (Ref. 10): the Mach disk forms such that its position is compatible with the reaccelerating transonic flow downstream of the disk. Abbett was the first to develop a fairly rigorous analysis of Mach disk formation. The flow upstream of the Mach disk is computed with the rotational method of characteristics. Downstream of the Mach disk, the flow outside of the slip line (Fig. 1) is also computed with the method of characteristics. The slip line is assumed to be inviscid and the flow between the slip line and the axis is assumed to be one-dimensional and isentropic. A trial position of the Mach disk is assumed, and the combined supersonic-subsonic flow downstream of the disk is computed. If the Mach disk is assumed too far upstream, then the slip line will reach a minimum radius, a "throat", before the flow on the axis reaches sonic speed. If the Mach disk is assumed too far downstream, then the flow on the axis will become sonic before the slip line radius reaches a minimum. The position of the Mach disk is iterated until the flow on the axis becomes sonic at a station where the slip line forms a throat.

The throat-like condition downstream of the Mach disk is a saddle point singularity in the flow-field solution. As usual, with saddle-point singularities, it is not too difficult to bracket the position of the Mach disk to acceptable uncertainty. But it is very difficult to establish the Mach disk position precisely enough so that the downstream solution will pass smoothly through the transonic region. Even if the Mach disk position is bracketed very tightly, to one part in 10^5 or 10^6, the solution will in general not pass smoothly through the transonic region. However, it is possible to use extrapolation techniques

to approximate the solution through the transonic region. Once the plume flow field downstream of the Mach disk is fully supersonic, the Abbett method can, in principle, be used to compute the second cycle of the plume shock structure. However, the assumption of one-dimensional flow behind the Mach disk may be less valid for the second Mach disk than for the first.

Abbett presented the solution for only one plume flow field. However, Fox, who developed a plume shock structure analysis which is essentially the same as Abbett's, has demonstrated that the method yields accurate predictions of the Mach disk location for a variety of nozzle exit Mach numbers and pressure ratios between 3 and 16 (Ref. 11).

Using the concept that the Mach disk is uniquely related to the transonic flow downstream of the disk, Peters and Phares (Ref. 8) developed a plume shock structure analysis which is much simpler than Abbett's. Instead of iteratively calculating the flow field downstream to the throat region, the generalized shape of the slip line downstream of the triple point was assumed. A third triple point boundary condition (in addition to $p_3 = p_4$ and $\theta_3 = \theta_4$) was written:

$$dp/dx|_3 = dp/dx|_4 \tag{1}$$

That is, the slip line just downstream of the triple point must allow the flow on either side of the slip line to develop compatibly. The flow upstream of the Mach disk is computed with the rotational method of characteristics. At each axial location in the boundary shock computation, a triple point computation is made to define p_3 and θ_3 (Fig. 2). By assuming that the slip line is a circular arc, $dp/dx|_4$ can be computed, based on only the local properties. Linearized supersonic flow theory is used to compute $dp/dx|_3$. The Mach disk position is established when Eq. (1) is satisfied.

Although based on a sound physical mechanism, the Peters and Phares model yields only fair predictions of the experimental Mach disk position (Ref. 8). The failure of the model can be attributed to two causes. First, the flow outside of the slip line (region 3 of Fig. 2) is highly rotational and the assumed adequacy of linearized theory is incorrect. Second, the assumed circular arc slip line, from the triple point to the sonic station, is incorrect. Experiments (e.g., Ref. 12) have shown that the slip line radius downstream of the triple point first increases to a maximum before turning toward the axis. The Abbett analysis predicts such a complex slip line geometry.

The analysis described in this report is basically similar to that described in Ref. 8, in that the coupling between the Mach disk and the transonic region is described in terms of the third triple point boundary condition (Eq. 1). However, the description of the slip line geometry has been improved, and the assumption of linearized supersonic flow in region 3 has been discarded. Physically realistic results are obtained without the time-consuming flow-field iterations of the Abbett analysis.

2.0 DEVELOPMENT OF ANALYTICAL MODEL

The approach taken in the present analysis has been to use the Fox version of the Abbett analysis to compute the slip line geometries for a series of plume flows. "Empirical" equations were then developed to correlate the results of the Abbett analysis for the slip line geometrical parameters. The empirical slip line parameters are used, in conjunction with a single rotational method of characteristics field point calculation in region 3 of Fig. 2, to evaluate the validity of each trial position of the triple point.

The slip line geometries for 18 plume flows were computed with the Fox-Abbett computer program. The 18 flows represented a random

sampling for M_n from 1.0 to 2.5, p_n/p_∞ from 3 to 16, γ from 1.2 to 1.4, and nozzle exit half-angles from 0 to 15 deg. (In all cases, the nozzle exit flow was assumed to be source-like). Two characteristic length scales for the slip line were identified (Fig. 3): (1) the distance, ℓ_1, from the Mach disk to the station where the slip line radius is a maximum, r_{smax} and (2) the distance, ℓ_2, from the Mach disk to the sonic point on the axis.

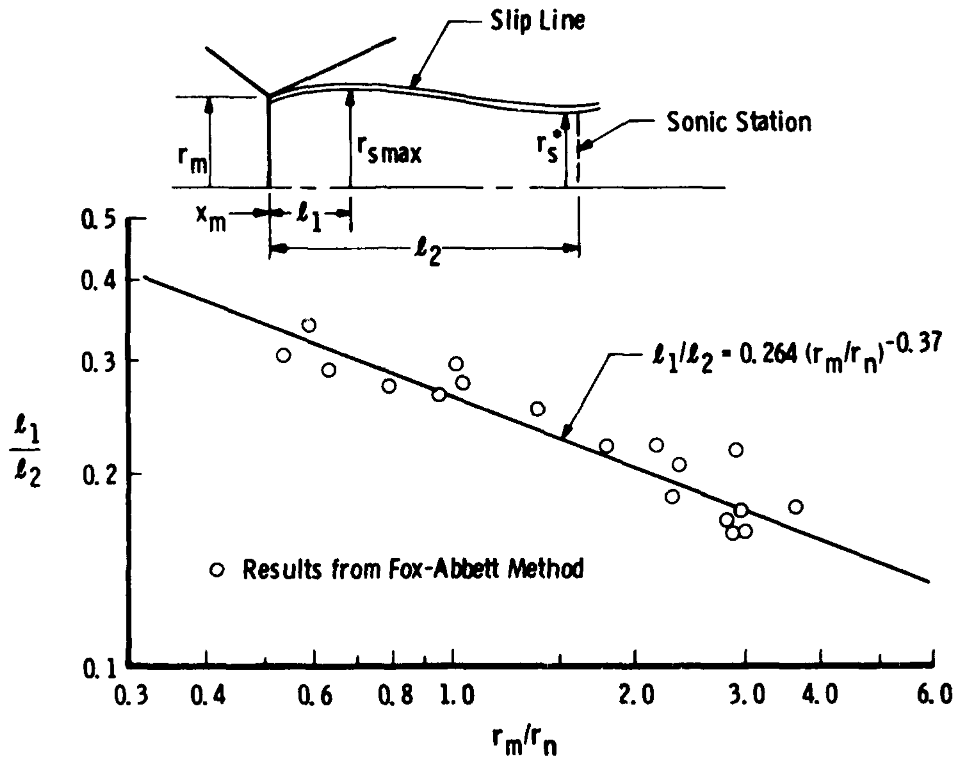


Figure 3. Ratio of length scales for the slip line.

The ratio of ℓ_1 to ℓ_2 was found to correlate well as a function of the Mach disk radius, r_m/r_n. As shown in Fig. 3, the Fox-Abbett results are well described by

$$\ell_1/\ell_2 = 0.264\ (r_m/r_n)^{-0.37} \qquad (2)$$

The ratio ℓ_2/r_m was found to correlate well with the flow angle upstream of the triple point, θ_1. As shown in Fig. 4, the Fox-Abbett results are well described by

$$\ell_2/r_m = 130(\theta_1)^{-1.55} \tag{3}$$

Equations (2) and (3) are combined to yield

$$\ell_1/r_m = 34.3(r_m/r_n)^{-0.37}(\theta_1)^{-1.55} \tag{4}$$

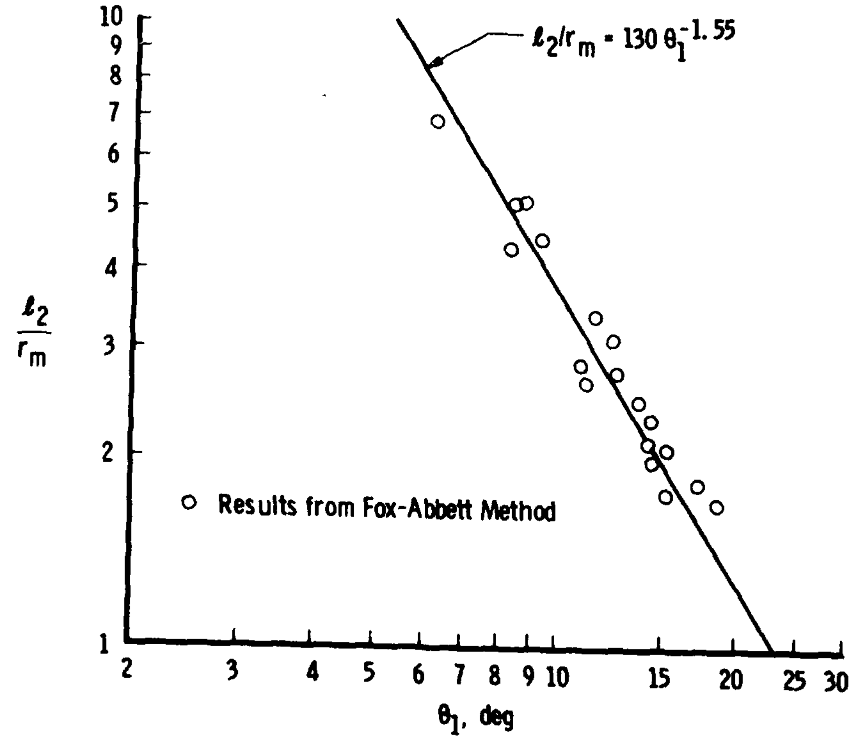


Figure 4. Correlation of the length scale ℓ_2.

The slip line geometry between x_m and $(x_m + \ell_1)$ is approximately parabolic (Fig. and is described by

$$\left(\frac{r_s - r_m}{r_{smax} - r_m}\right) = 2\left(\frac{x - x_m}{\ell_1}\right) - \left(\frac{x - x_m}{\ell_1}\right)^2 \tag{5}$$

But $\tan\theta_4 = 2\,(r_{smax} - r_m)/\ell_1$, so that

$$r_s = r_m + (\tan\theta_4)(x - x_m) - \frac{1}{2}(\tan\theta_4)(x - x_m)^2/\ell_1 \qquad (6)$$

and

$$r_{smax} = r_m + \frac{1}{2}\,\ell_1 \tan\theta_4 \qquad (7)$$

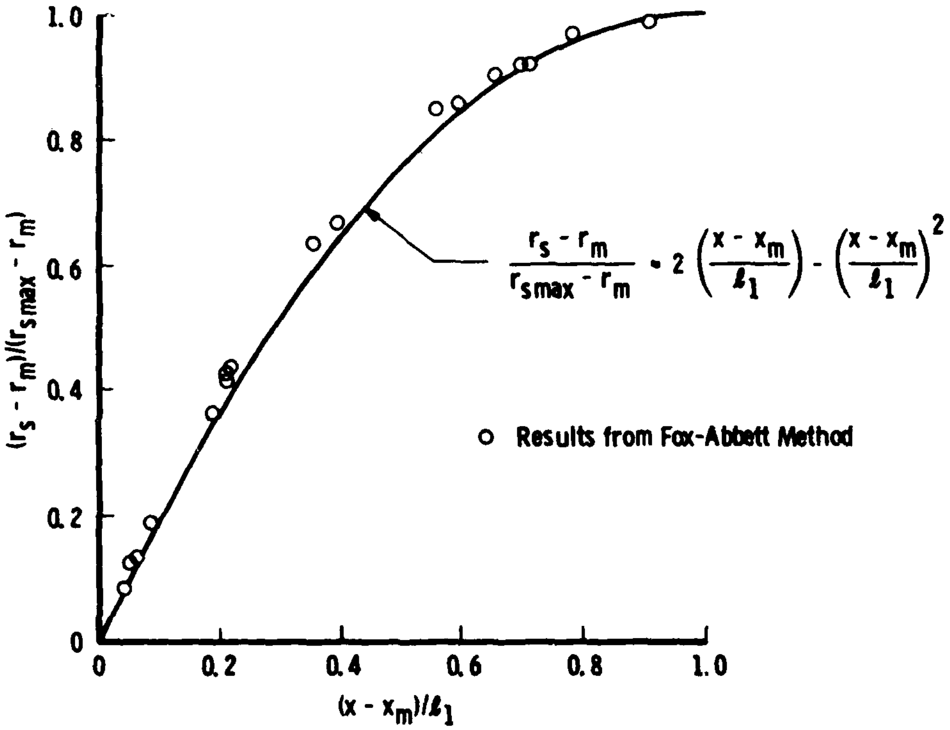


Figure 5. Parabolic slip line geometry between $x = x_m$ and $x = x_m + \ell_1$.

Downstream of the maximum slip line radius, the slip line shape (Fig. 6) is approximately described by

$$\left(\frac{r_s - r_s^*}{r_{smax} - r_s^*}\right) = \left[\frac{1}{2} + \frac{1}{2}\cos\left(\pi\,\frac{x - x_{max}}{x^* - x_{max}}\right)\right]^{0.67} \qquad (8)$$

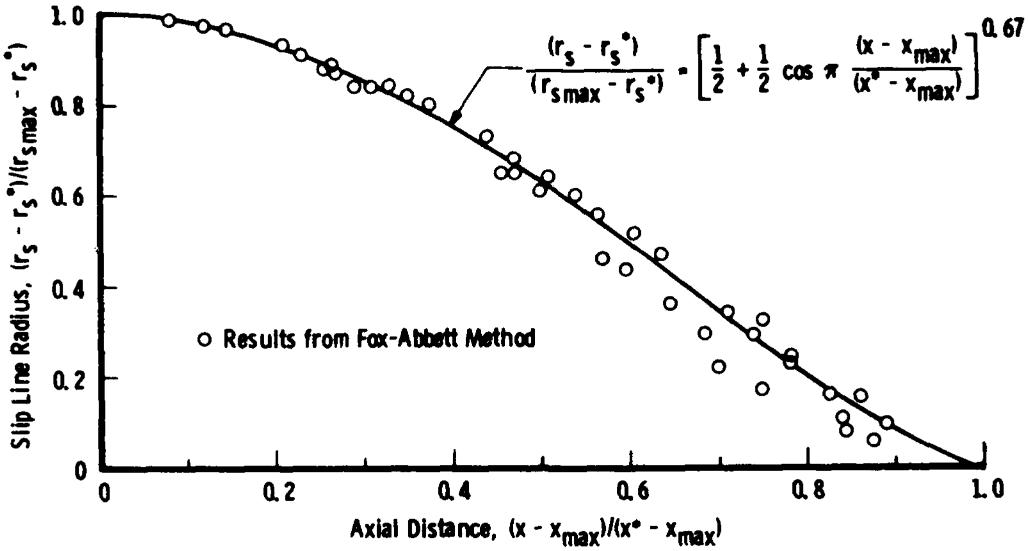


Figure 6. Slip line geometry between the maximum radius station and the sonic station.

<u>Triple point computation procedure</u> - The flow field upstream of the Mach disk is computed with the rotational method of characteristics. The computer program is an extension of the "2-D Core Theory" of Refs. 13 and 14, which includes the capability for a fully coupled computation of the turbulent mixing layer along the plume boundary. However, in order to consistently compare the present model with the inviscid Fox-Abbett model, the turbulent mixing at the plume boundary was suppressed in most of the computations presented in this report.

A trial triple point is assumed at each axial station in the computation of the boundary shock wave. The Rankine-Hugoniot equations are used, along with the triple point boundary conditions ($p_3 = p_4$, $\theta_3 = \theta_4$), to compute p_4 and θ_4. The values of M_1 and θ_1 upstream of the trial triple point are used to compute ℓ_1 from Eq. (4). Then Eq. (6) is used to establish the geometry of the slip line. With the known slip line geometry and the assumption that the subsonic flow is one-dimensional and isentropic, the pressure distribution and flow angle distribution along the slip line are established.

From a point "a" just downstream of the triple point (Fig. 7), at which the pressure and flow angle are known, a left running characteristic is generated to establish the coordinates and flow properties of point "b" just behind the reflected shock wave. Then the location of point "c" on the slip line is computed with the equation which describes the geometry of the right running characteristic between points "b" and "c"; the axial distance between x_m and point "c" is typically only a few percent of ℓ_1. All flow properties at point "c" are fully defined by the equations which describe the flow in the one-dimensional streamtube. If the compatibility equation for the right running characteristic between points "b" and "c" is not satisfied, then the assumed slip line geometry does not permit consistent development of the flow on either side of the slip line, and Eq. (1) is not satisfied. Therefore, the computation of the flow field is continued downstream, with a trial triple point assumed at the next axial station. At some axial station, the compatibility equation for the right running characteristic will be satisfied; this station is the Mach disk location.

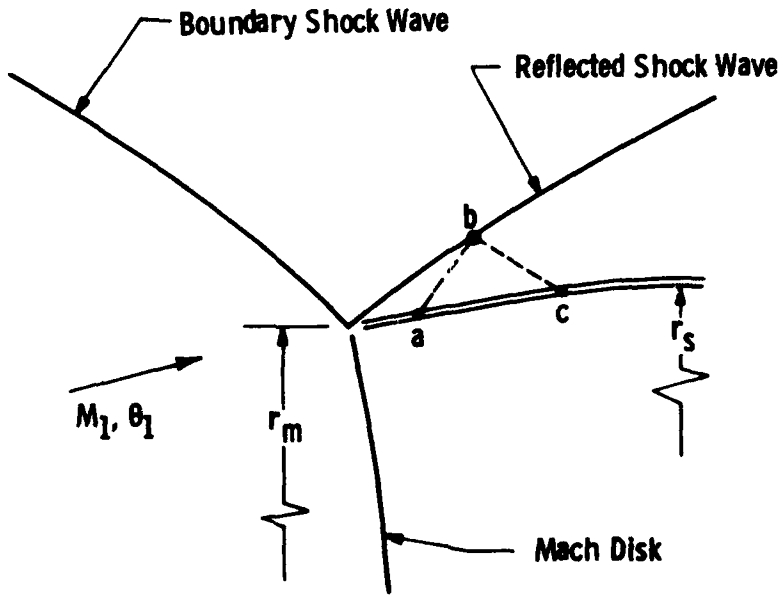


Figure 7. Triple point computation procedure.

Computation procedure downstream of the Mach disk - After the Mach disk position is established, the flow-field computation can be continued downstream beyond the sonic station on the axis. Equations (3) and (4) are used to compute ℓ_1 and ℓ_2, r_{smax} is computed from Eq. (7), and $r_s{}^*$ is computed from the isentropic area ratio equation. Equation (6) describes the slip line geometry between x_m and $(x_m + \ell_1)$, and Eq. (8) describes the slip line geometry between $(x_m + \ell_1)$ and $(x_m + \ell_2)$.

With the inviscid slip line geometry prescribed, the rotational method of characteristics is used to compute the supersonic flow outside of the slip line. The computation is continued downstream until the static pressure on the slip line is slightly less than the sonic pressure for the one-dimensional streamtube. Because the supersonic flow field is generated along a prescribed geometry (the slip line), no singularity appears in the solution.

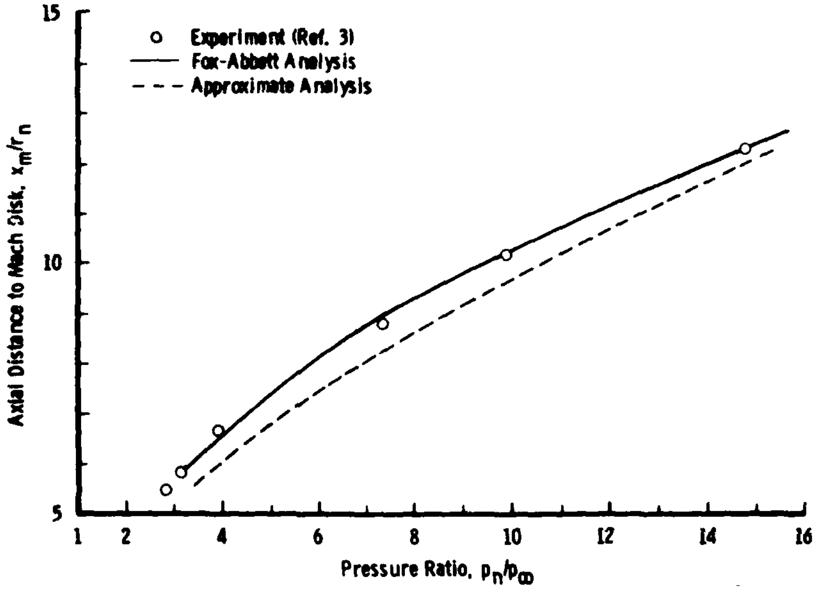
3.0 DISCUSSION OF RESULTS

The Mach disk locations predicted with the approximate plume shock structure model are compared with the experimental results of Love, et al., (Ref. 3) in Fig. 8. The nozzle air flow exhausted into quiescent surroundings. All of the results to be discussed were obtained with nozzles which were designed to provide uniform and parallel flow at the exit plane.

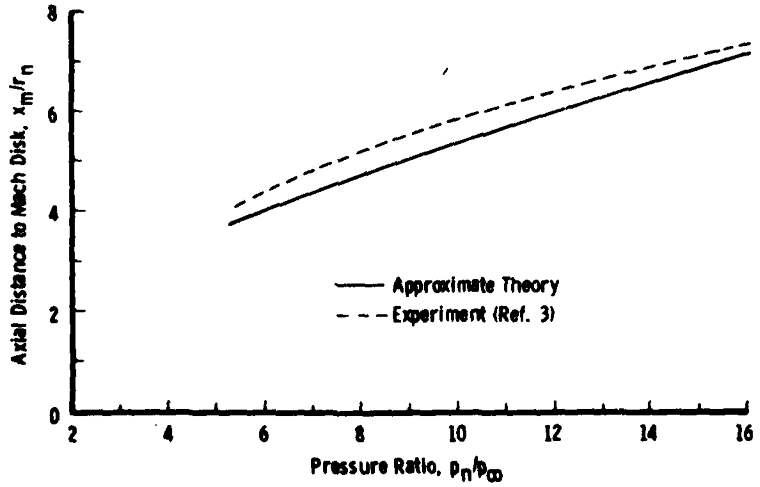
Computed and experimental Mach disk positions are shown in Fig. 8a for $M_n = 2.0$. The Mach disk positions predicted by the approximate analysis are about ten-percent upstream of the experimental positions at the lower values of p_n/p_∞, but only a few percent upstream of experiment for $p_n/p_\infty = 15$. The Fox-Abbett analysis provides excellent predictions of the experimental Mach disk locations.

Computed and experimental Mach disk locations are shown in Figs. 8b, c, and d for M_n of 1.0, 1.5, and 2.5, respectively. In addition to

the experimental results of Love, et al., for $M_n = 2.5$, two experimental points obtained at the AEDC for $M_n = 2.44$ are shown in Fig. 8d. In general, the predictions of the approximate model are no more than ten percent in error at the low-pressure ratios, and the predictions improve as p_n/p_∞ increases toward 16.

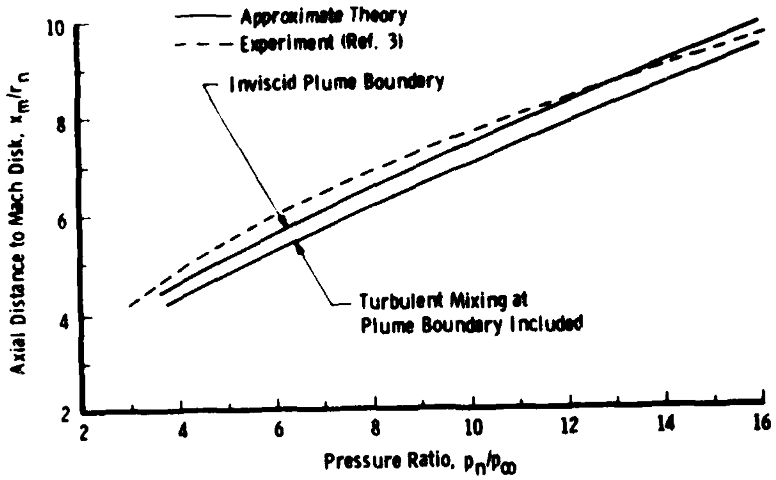


a. $M_n = 2.0$

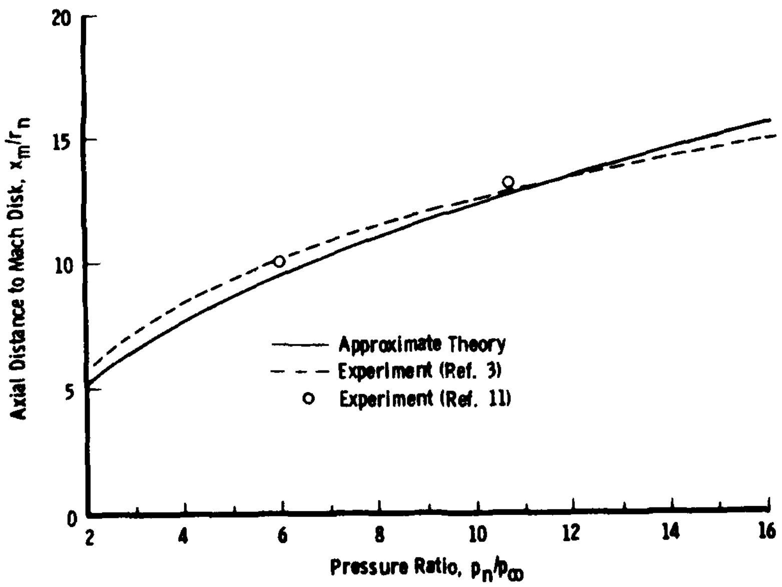


b. $M_n = 1.0$

Figure 8. Prediction of the axial location of Mach disk.



c. $M_n = 1.5$



d. $M_n = 2.5$

Figure 8. Concluded.

Very little information is available on the flow structure downstream of the Mach disk. R. A. Paulk and C. E. Peters conducted a simple experiment to define the sonic station downstream of the Mach

disk in the plume from a Mach number 2.44 nozzle operated at p_n/p_∞ = 10.7 (Ref. 11). Blunt bodies of various sizes were placed on the plume axis downstream of the Mach disk. The bodies were then translated upstream toward the Mach disk. At some critical distance between the Mach disk and the body, the Mach disk was observed, with a shadowgraph, to be affected by the body. By extrapolating the critical distance function to zero body size (a negligible disturbance), the axial distance, ℓ_2, between the Mach disk and the sonic station was established. The experimental results, as well as the predictions of the Fox-Abbett and approximate analyses, are shown below.

	x_m/r_n	ℓ_2/r_n
Experiment	13.1	6.4
Fox-Abbett Analysis	13.06	6.44
Approximate Analysis	12.35	6.45

Both predicted ℓ_2 values are in excellent agreement with the experimental value.

In the computations which have been discussed up to this point, the turbulent mixing at the plume boundary has been neglected. Inclusion of the turbulent mixing causes the boundary shock wave to be stronger than in the inviscid boundary case. As a result, the Mach disk is predicted to occur upstream of the location predicted for an inviscid plume boundary. However, as shown in Fig. 8c for M_n = 1.5, inclusion of the boundary mixing changes the predicted value of x_m by only about five percent.

The results which are presented in Figs. 8a through d indicate that the approximate plume model yields satisfactory predictions of the Mach disk location for p_n/p_∞ between 3 and 16, that is, satisfactory results are obtained over the p_n/p_∞ range which was used to develop the empirical slip line geometrical parameters. However, only fair results

are usually obtained for p_n/p_∞ values of 1.5 to 2.5. The empirical slip line functions must be refined if the approximate model is to yield good results for low-pressure ratios. The validity of the empirical slip line functions at pressure ratios greater than 16 was not investigated.

Because the iterative flow-field computations of the Abbett method have been eliminated in the approximate shock structure model, relatively small computation times are required. Typically, computation of the first cycle of a plume requires one minute on an IBM 370/165 digital computer. The computation time is not significantly affected by including the turbulent mixing along the plume boundary.

4.0 CONCLUDING REMARKS

The main objective of this study, to develop a computationally efficient and reasonably accurate analytical model for the first cycle of the plume shock structure, has been achieved. However, acceptably accurate predictions of the Mach disk location can be expected only for p_n/p_∞ between 3 and 16, the range for which the slip line geometrical correlations were developed. These correlations must be improved if the Mach disk location is to be accurately predicted for other pressure ratios. Until improved correlation equations are available, it is suggested that, at pressure ratios less than 3, the Mach disk location be established with the Lewis and Carlson empirical equation. The techniques of the present study can then be used to compute the flow through the transonic region downstream of the Mach disk.

In principle, the present analytical model, as well as the Abbett model, is applicable to the computation of the second cycle of the plume shock structure. However, the flow approaching the second Mach disk is much more rotational than the flow approaching the first disk. Therefore, the assumption of one-dimensional flow behind the second

Mach disk may not be acceptably accurate. In addition, computation of the second cycle should probably include the mixing along the slip line which extends downstream from the first Mach disk. Clearly, the applicability of presently available models to the computation of the second cycle of the plume requires further study. However, further study will be hampered by the lack of detailed experimental information on the flow structure downstream of the first Mach disk.

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NOMENCLATURE

ℓ_1	Axial distance from the Mach disk to the station where the slip line radius is a maximum
ℓ_2	Axial distance from the Mach disk to the station where flow on the axis is sonic
M	Mach number
M_n	Nozzle exit Mach number
p	Static pressure
p_n	Nozzle exit static pressure
p_∞	Ambient pressure
r_m	Mach disk radius
r_n	Nozzle exit radius
r_s	Slip line radius
r_{smax}	Maximum slip line radius
$r_s{}^*$	Slip line radius at the sonic station
x	Axial coordinate
x_m	Axial distance from the nozzle exit plane to the Mach disk

x_{max} Axial distance from the nozzle exit plane to the station where the slip line radius is a maximum, $= x_m + \ell_1$

x^* Axial distance from the nozzle exit plane to the station where the flow on the axis is sonic, $= x_m + \ell_2$

γ Ratio of specific heats

θ Flow angle

SUBSCRIPTS

1,2,3,4 Flow regions defined in Fig. 2